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A TRACING LOAD IN A NONLINEAR PLANE ELASTIC PROBLEM

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We consider a plane problem of elasticity theory to determine the features of allowance for a tracing load (normal pressure) in a geometrically nonlinear approach. The notation and terminology used are the same as in [1–4].

1. For a plane homogeneous problem of nonlinear elasticity theory with stresses specified on the contour of a region (Fig. 1) the following simple compact relations hold [1–4]:

$$\frac{\partial\{F^{-1}\cdot J\Sigma\}_1}{\partial\bar{\zeta}}+\frac{\partial\{F^{-1}\cdot J\Sigma\}_2}{\partial\zeta}=0 \qquad (\zeta\in\mathring{V}); \tag{1.1}$$

$$\{F^{-1}\cdot J\Sigma\}_1=\frac{\partial z}{\partial\zeta}\left|\frac{\partial z}{\partial\zeta}\right|^{-1}\frac{\partial\Phi}{\partial|\partial z/\partial\zeta|} \qquad (\zeta\in\mathring{V}); \tag{1.2$_1$}$$

$$\{F^{-1}\cdot J\Sigma\}_2=\frac{\partial z}{\partial\bar{\zeta}}\left|\frac{\partial z}{\partial\bar{\zeta}}\right|^{-1}\frac{\partial\Phi}{\partial|\partial z/\partial\bar{\zeta}|} \qquad (\zeta\in\mathring{V}); \tag{1.2$_2$}$$

$$\{F^{-1}\cdot J\Sigma\}_2 e^{i\mathring{\gamma}}+\{F^{-1}\cdot J\Sigma\}_2 e^{-i\mathring{\gamma}}=2e^{i\mathring{\gamma}}\left[\sigma_{\mathring{\nu}\mathring{\nu}}(\mathring{s})+i\sigma_{\mathring{\nu}\mathring{t}}(\mathring{s})\right] \qquad (\zeta\in\mathring{S}). \tag{1.3}$$

Here $\{F^{-1}\cdot J\Sigma\}_i$ are the complex components of the nonsymmetric tensor of nominal stresses; $\zeta=\mathring{x}_1+i\mathring{x}_2$ and $z=x_1+ix_2$ are the complex coordinates of a material point before and after deformation; $\Phi(|\partial z/\partial\zeta|,|\partial z/\partial\bar{\zeta}|,\lambda)$ is the elastic potential; and λ is the extension ratio in the direction of the third coordinate axis $\mathring{x}_3$.

Using the basic functions $\partial z/\partial\zeta$ and $\partial z/\partial\bar{\zeta}$ determined from the boundary problem (1.1)–(1.3), we find the rotation of the material particle ω and the multiplicity of area variations Δ

$$e^{i\omega}=\frac{\partial z}{\partial\zeta}\left|\frac{\partial z}{\partial\zeta}\right|^{-1}, \qquad \Delta=\left|\frac{\partial z}{\partial\zeta}\right|^2-\left|\frac{\partial z}{\partial\bar{\zeta}}\right|^2,$$

the complex components of the conditional stress tensor (symmetric Biot tensor)

$$\mathring{\Sigma}_1\equiv\mathring{\sigma}_{11}+\mathring{\sigma}_{22}=\frac{\partial\Phi}{\partial|\partial z/\partial\zeta|}, \qquad \mathring{\Sigma}_2\equiv\mathring{\sigma}_{11}-\mathring{\sigma}_{22}+i2\,\mathring{\sigma}_{12}=\frac{\partial\bar{z}}{\partial\zeta}\frac{\partial z}{\partial\bar{\zeta}}\left|\frac{\partial z}{\partial\zeta}\right|^{-1}\left|\frac{\partial z}{\partial\bar{\zeta}}\right|^{-1}\frac{\partial\Phi}{\partial|\partial z/\partial\bar{\zeta}|},$$

and the complex coordinate of the material point after deformation

$$z=\int\left(\frac{\partial z}{\partial\zeta}d\zeta+\frac{\partial z}{\partial\bar{\zeta}}d\bar{\zeta}\right).$$

2. To the elastic potential

$$\Phi=\sigma^*\left|\frac{\partial z}{\partial\zeta}\right|^2+\alpha\left|\frac{\partial z}{\partial\bar{\zeta}}\right|^2 \tag{2.1}$$

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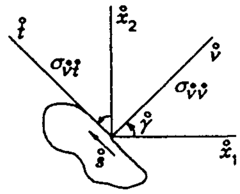

Fig. 1

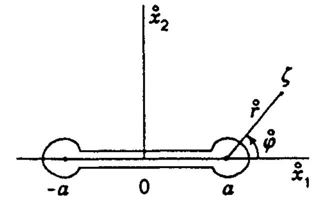

Fig. 2

corresponds, in the principal coordinate axes of strain, the following relation between the principal conditional stresses $\mathring{\sigma}_i$ and the principal extension ratios:

$$\mathring{\sigma}_1 = \sigma^*\{1 + (1/2)[(\lambda_1 - 1) + (\lambda_2 - 1)]\} + (1/2)\alpha[(\lambda_1 - 1) - (\lambda_2 - 1)],$$
$$\mathring{\sigma}_2 = \sigma^*\{1 + (1/2)[(\lambda_1 - 1) + (\lambda_2 - 1)]\} - (1/2)\alpha[(\lambda_1 - 1) - (\lambda_2 - 1)].$$

When $\lambda_1 = \lambda_2 = 1$, i.e., in the absence of strain, we have $\mathring{\sigma}_1 = \mathring{\sigma}_2 = \sigma^*$. Thus, σ^* is the preliminary, conditional, uniform (in the plane $\mathring{x}_1\mathring{x}_2$) tensile stress. Expression (2.1) shows that elastic potential (2.1) corresponds to prestressed (physically) linear material. Note the relationship between the elastic constants σ^* and α with the traditional ones:

$$\sigma^* \to \frac{E}{(1+\nu)(1-2\nu)}, \qquad \alpha \to \frac{E}{1+\nu}. \tag{2.2}$$

To the given material corresponds

$$\frac{\partial z}{\partial \zeta} = \Phi(\zeta), \qquad \frac{\partial z}{\partial \bar{\zeta}} = \overline{\Psi(\zeta)}, \quad . \; z = \int \Phi(\zeta)d\zeta + \overline{\int \Psi(\zeta)d\zeta}.$$

In this case, for an infinite region with an opening that contains the coordinate origin, we have

$$\Phi(\zeta) = a_0 + \Phi_0(\zeta), \quad \Phi_0(\zeta) = \frac{a_{-2}}{\zeta^2} + \frac{a_{-3}}{\zeta^3} + \cdots, \quad \Psi(\zeta) = b_0 + \Psi_0(\zeta), \quad \Psi_0(\zeta) = \frac{b_{-2}}{\zeta^2} + \frac{b_{-3}}{\zeta^3} + \cdots,$$

where

$$a_0 = (\mathring{\sigma}_{11}^{\infty} + \mathring{\sigma}_{22}^{\infty})/2\sigma^*; \qquad b_0 = (\mathring{\sigma}_{11}^{\infty} - \mathring{\sigma}_{22}^{\infty} - i2\,\mathring{\sigma}_{12}^{\infty})/2\alpha;$$

and $\mathring{\sigma}_{ij}^{\infty}$ are the conditional stresses at infinity.

3. As a base problem, we consider the standard problem of a plane with a linear cut ($-a \leqslant \mathring{x}_1 \leqslant a$, $\mathring{x}_2 = 0$) that is stretched in all directions at infinity (Fig. 2), so that $\mathring{\sigma}_{11}^{\infty} = \mathring{\sigma}_{22}^{\infty} = \sigma^*$, $\mathring{\sigma}_{12}^{\infty} = 0 \to a_0 = 1$, and $b_0 = 0$. In addition, the cut edges are subjected to a uniform normal pressure σ_0. Three variants should be distinguished in this case:

(1) The tracing load (the normal pressure σ_0) traces the normal to the deformed cut sides. In this case, static boundary condition (1.3) takes the form

$$(\sigma^* + \sigma_0)\Phi_0(\zeta)e^{i\mathring{\gamma}} + (\alpha - \sigma_0)\overline{\Psi}(\zeta)e^{-i\mathring{\gamma}} = -(\sigma^* + \sigma_0)e^{i\mathring{\gamma}}. \tag{3.1}$$

Here σ_0, entering into the boundary-condition coefficients, is a peculiar parametric load. It is important [and this is a characteristic feature of elastic potential (2.1)] that Eq. (3.1) is linear with respect to Goursat–Kolosov functions (with substantial nonlinearity of the general problem). Any method used in the linear theory can be applied to this equation. Moreover, this boundary condition is even simpler than its linear analog.

Solution of Eq. (3.1) shows that the cut becomes a circle with radius

$$R = \frac{R_0}{1 - \sigma_0/\alpha}, \qquad R_0 = \frac{(1 + \sigma^*/\alpha)a}{2}, \tag{3.2}$$

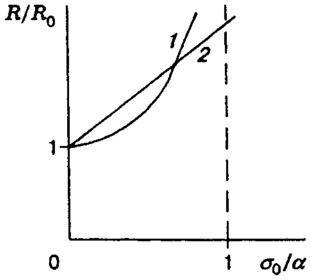

Fig. 3

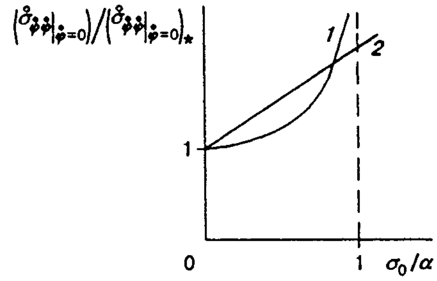

Fig. 4

which is qualitatively confirmed by experiments with rubber plates. The asymptotic behavior of the stress state near the right end of the cut is determined by the quantity

$$\overset{\circ}{\sigma}_{\overset{\circ}{\varphi}\overset{\circ}{\varphi}}\Big|_{\overset{\circ}{\varphi}=0} = \left(\overset{\circ}{\sigma}_{\overset{\circ}{\varphi}\overset{\circ}{\varphi}}\Big|_{\overset{\circ}{\varphi}=0}\right)_* \frac{2+(\alpha/\sigma^*-1)(\sigma_0/\alpha)}{2[1-(\sigma_0\alpha)]}, \qquad \left(\overset{\circ}{\sigma}_{\overset{\circ}{\varphi}\overset{\circ}{\varphi}}\Big|_{\overset{\circ}{\varphi}=0}\right)_* = \sigma^*\left(\frac{a}{2\overset{\circ}{r}}\right)^{1/2}).$$

(2) In a geometrically nonlinear approach we also assume that $\sigma_{\overset{\circ}{\nu}\overset{\circ}{\nu}}(\overset{\circ}{s}) = -\sigma_0$, $\sigma_{\overset{\circ}{\nu}\overset{\circ}{t}}(\overset{\circ}{s}) = 0$, i.e., that σ_0 is a constant ("dead") load which is normal to the sides of the undeformed cut. Note that such a load is realized in a region of weakened bonds (in the formation of so-called tension bars in polymers). In this case, boundary condition (1.3) takes the form

$$\sigma^*\Phi_0(\zeta)e^{i\overset{\circ}{\gamma}} + \alpha\overline{\Psi_0(\zeta)}e^{-i\overset{\circ}{\gamma}} = -(\sigma^*+\sigma_0)e^{i\overset{\circ}{\gamma}}.$$

The cut becomes a circle with radius

$$R = R_0\left[1+\frac{\alpha}{\sigma^*}\left(\frac{\sigma_0}{\alpha}\right)\right], \qquad R_0 = \frac{(1+\sigma^*/\alpha)a}{2}. \tag{3.3}$$

The asymptotic behavior of the stress state is determined by the quantity

$$\overset{\circ}{\sigma}_{\overset{\circ}{\varphi}\overset{\circ}{\varphi}}\Big|_{\overset{\circ}{\varphi}=0} = \left(\overset{\circ}{\sigma}_{\overset{\circ}{\varphi}\overset{\circ}{\varphi}}\Big|_{\overset{\circ}{\varphi}=0}\right)_*\left[1+\frac{\alpha}{\sigma^*}\left(\frac{\sigma_0}{\alpha}\right)\right], \qquad \left(\overset{\circ}{\sigma}_{\overset{\circ}{\varphi}\overset{\circ}{\varphi}}\Big|_{\overset{\circ}{\varphi}=0}\right)_* = \sigma^*\left(\frac{a}{2\overset{\circ}{r}}\right)^{1/2}.$$

This case can be regarded as corresponding to an initially normal load.

(3) Using the relations of linear theory (taking into account preliminary uniform extension), we obtain an expression for the asymptotic behavior of the stress state. Thus, for an initially normal load, linear theory gives a correct asymptotic behavior. Concerning the shape of the deformed cut, the obtained result is unsuitable for its description (the horizontal cut becomes vertical with penetration of the sides). This should be expected, because linear theory inadequately describes great rotations.

We take a value of $\nu = 0.3$ (which is usual for metals), for which, according to (2.2), $\alpha/\sigma^* = 1-2\nu = 0.4$. For these elastic constants, Figs. 3 and 4 show curves of R/R_0 [see (3.2) and (3.3)] and $\overset{\circ}{\sigma}_{\overset{\circ}{\varphi}\overset{\circ}{\varphi}}\big|_{\overset{\circ}{\varphi}=0} \Big/ \left(\overset{\circ}{\sigma}_{\overset{\circ}{\varphi}\overset{\circ}{\varphi}}\big|_{\overset{\circ}{\varphi}=0}\right)_*$ versus σ_0/α (curves 1 and 2 correspond to cases 1 and 2).

Note that this type of dependence on the normal pressure also holds for the angular points (cuts).

Thus, using the standard problem, we obtained (by linear and geometrical approaches) exact expressions for conditional and linear strains. Comparison of the obtained solutions showed a considerable difference between the effects of tracing and dead loads on the asymptotic behavior of strains and the shape of the deformed contour. The applicability of the asymptotic behavior of strains in linear theory in the case of dead loads was demonstrated.

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REFERENCES

1. K. F. Chernykh, *Nonlinear Elasticity Theory in Machine-Building Calculations*, Mashinostroenie, Leningrad (1986).
2. K. F. Chernykh, *Introduction to Anisotropic Elasticity*, Nauka, Moscow (1988).
3. K. F. Chernykh, "The nonlinear plane theory of elasticity and its application to physically and geometrically nonlinear fracture mechanics," *Usp. Mekh.*, **12**, No. 4, 51–75 (1986).
4. K. F. Chernykh, "On the necessity of taking into account geometrical nonlinearity in problems of brittle fracture," *Dokl. Akad. Nauk SSSR*, **306**, No. 6, 1336–1338 (1989).